AQA Core 3 Algebra and functions

Section 1: Functions

Notes and Examples

These notes contain subsections on:

The language of functions

You need to know the terminology associated with mappings and functions:

A **mapping** is any rule which associates two sets of items. In a mapping, the **input**, or **object** is something which is to be mapped to something else (the **output**, or **image**).

The set of all possible objects (inputs) of a mapping is called the **domain** of the mapping. The set of outputs for a particular set of inputs for a mapping is called the **range**.

There are four different types of mapping:

- A **one-to-one mapping** is a mapping in which each object is mapped to exactly one image, and each image is the image of exactly one object.

- A **one-to-many mapping** is a mapping in which an object may be mapped to two or more different images.

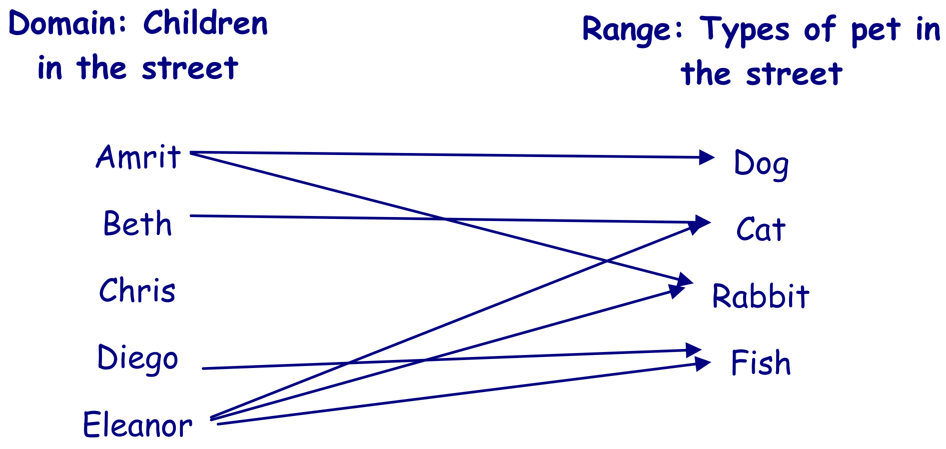
- A **many-to-one mapping** is a mapping in which two or more particular objects may be mapped to the same image.

- A **many-to-many mapping** is a mapping in which an object may be mapped to two or more different images, and in which two or more objects may be mapped to the same image.

A **function** is a mapping in which there is only one possible image for each object. A function may be one-to-one or many-to-one.

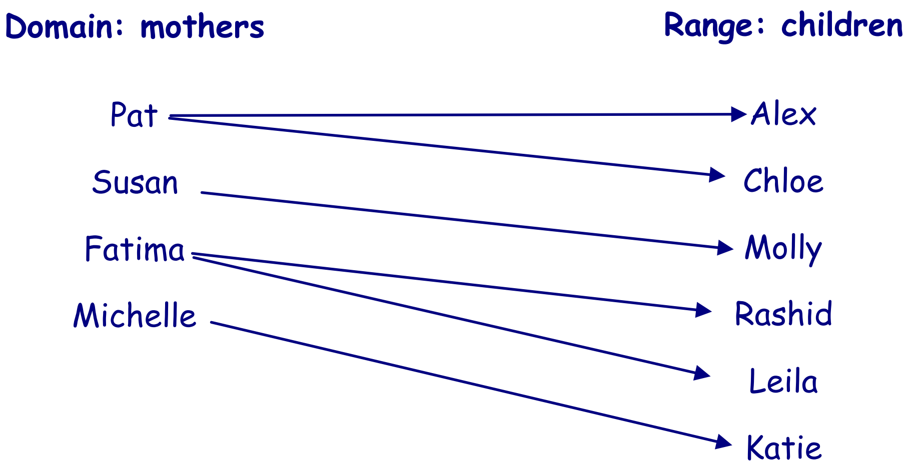
Here are some practical examples of the different types of mappings:

1. Mapping from the set of children who live in a particular street to the set of types of pet.



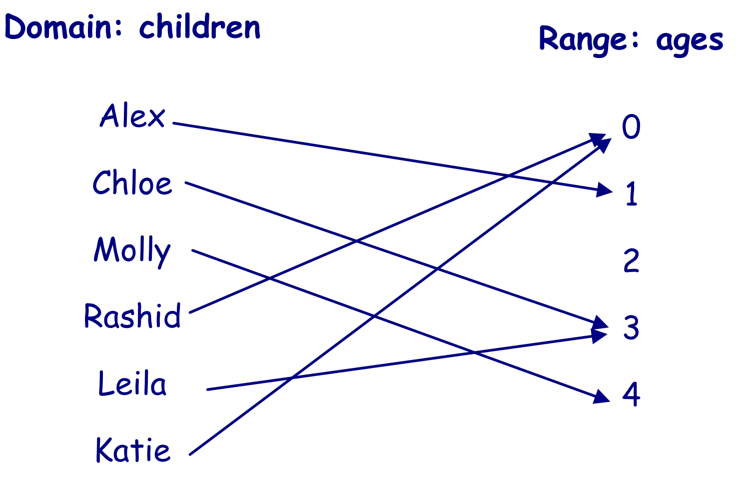
This mapping is many-to-many since a particular child may have more than one pet, and also a particular type of pet can be owned by more than one child.
As only four different types of pet are owned by this particular set of children, the range of the mapping is the set {dog, cat, rabbit, fish}.

2. Mapping from the set of mothers at a toddler group to the set of children at the group.



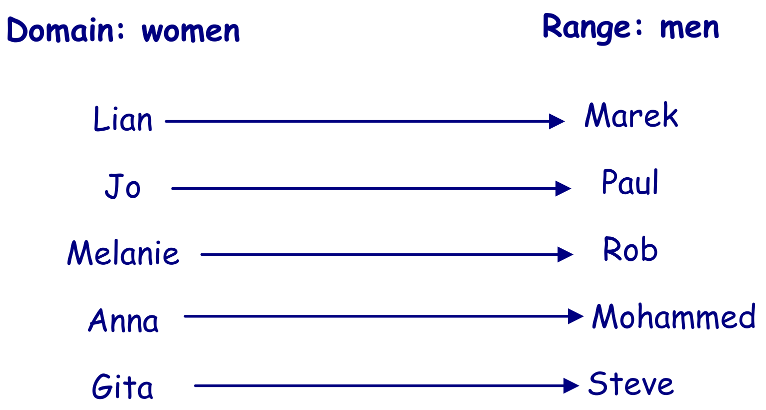
This is a one-to-many mapping: a particular mother may have more than one child, but a particular child has only one mother.

3. Mapping from the set of children at a toddler group to their ages.



This is a many-to-one mapping: a particular child has only one age, but there may be more than one child of a particular age.
Although the set of possible ages for children attending a toddler group is {0, 1, 2, 3, 4}, there are no 2-year-olds in this particular group, so the range in this case is the set {0, 1, 3, 4}.

4. Mapping from the set of women at a ballroom dancing class to the set of their partners.



This is a one-to-one mapping: each woman has exactly one partner, and each man is the partner of exactly one woman.

The examples above can be helpful to familiarise yourself with the terminology associated with mappings, but in situations like these it is sometimes difficult to define the sets involved clearly and unambiguously (for example, what would you include in types of pet?). However, in mathematical mappings, sets can be defined precisely. Examples of possible domains include the set of integers, the set of real numbers, or a restricted range of values such as the set of real numbers x for which $0 < x < 1$.

AQA C3 Algebra 1 Notes and Examples

You may already be familiar with the mathematical "shorthand" that is often used to define particular sets of numbers. Here is a reminder of some of the notation which you may find useful:

$\in$	means "is an element of"
:	means "such that"
$\mathbb{Z}$	denotes the set of integers
$\mathbb{Q}$	denotes the set of rational numbers
$\mathbb{R}$	denotes the set of real numbers
$\mathbb{R}^+$	denotes the set of positive real numbers (similarly for rational numbers, integers etc.)

So, for example,

$x \in \mathbb{R}$	means that x is a real number
$y \in \mathbb{Z} : 0 < y < 10$	means that y is an integer such that y lies between 0 and 10 (not including either 0 or 10).

Here are some examples of mathematical mappings:

1. $x \to 1-3x, x \in \mathbb{Q}$
   This is a one-to-one mapping. For every value of x, there is one value of $1 - 3x$, and no two objects map to the same image.
   The range of this mapping is also $\mathbb{Q}$.
   This mapping is also a function as there is only one possible image for each object.

2. $x \to x^2 + 3, x \in \mathbb{R}$
   This is a many-to-one mapping, since each point has only one possible image, but, for example, -1 and 1 both have image 4.
   All image points are in $\mathbb{R}$, but the image points are all greater or equal to 3, so the range can be written as $y \in \mathbb{R} : y \geq 3$.
   This mapping is also a function as there is only one possible image for each object.

3. $x \to \pm\sqrt{x}, x \in \mathbb{Z}^+$
   This is a one-to-many mapping, since each object has two possible images, but no two objects map to the same image.
   The image points of this mapping are not all integers, but they are all real numbers. The range is not the whole of $\mathbb{R}$, since there are plenty of real numbers which are not the square root of an integer. The range can be written as $y \in \mathbb{R} : y^2 \in \mathbb{Z}$.
   This mapping is not a function as there is more one possible image for each object.

4. $x \to \pm\sqrt{x^2 + 1}, x \in \mathbb{R}$
   This is a many-to-many mapping, since each object has two possible

images (e.g. 1 maps to $\sqrt{2}$ and $-\sqrt{2}$) and two objects map to each image (e.g. 1 and -1 both map to $\sqrt{2}$).
The range of this mapping is $\mathbb{R}$.
This mapping is not a function as there is more one possible image for each object.

The Mathcentre video ***Introduction to functions*** covers definitions and terminology of functions.

Composite functions

The important thing to remember when finding a composite function is the order in which the functions are written: $\mathrm{fg}(x)$ means first apply the function g to x, then apply the function f to the result.

Example 1
The functions f, g and h are defined by:

$$\mathrm{f}(x) = x + 1$$
$$\mathrm{g}(x) = x^2$$
$$\mathrm{h}(x) = 3x$$

Find the following composite functions:
(i) $\mathrm{fg}(x)$ (ii) $\mathrm{gh}(x)$ (iii) $\mathrm{hgf}(x)$ (iv) $\mathrm{f}^2(x)$

Solution

(i) $\mathrm{fg}(x) = \mathrm{f}[\mathrm{g}(x)]$
$= \mathrm{f}(x^2)$
$= x^2 + 1$

Apply g followed by f;
i.e. square, then add 1.

(ii) $\mathrm{gh}(x) = \mathrm{g}[\mathrm{h}(x)]$
$= \mathrm{g}(3x)$
$= (3x)^2$
$= 9x^2$

Apply h followed by g;
i.e. multiply by 3, then square.

(iii) $\mathrm{hgf}(x) = \mathrm{hg}[\mathrm{f}(x)]$
$= \mathrm{h}[\mathrm{g}(x+1)]$
$= \mathrm{h}[(x+1)^2]$
$= 3(x+1)^2$

Apply f followed by g followed by h;
i.e. add 1, then square, then multiply by 3.

(iv) $\mathrm{f}^2(x) = \mathrm{f}[\mathrm{f}(x)]$

$$= \mathrm{f}(x+1)$$
$$= (x+1)+1$$
$$= x+2$$

Apply f twice;
i.e. add 1, then add 1.

Example 2
The functions f and g are defined as:

$$\mathrm{f}(x) = \frac{1}{x}$$
$$\mathrm{g}(x) = 2x$$

Write the functions:

(i) $\frac{1}{2x}$ (ii) $\frac{2}{x}$ (iii) $4x$

in terms of the functions f and g.

Solution

(i) This function is obtained by first multiplying by 2, then taking the reciprocal; i.e. applying g followed by f. So this function is fg.

(ii) This function is obtained by first taking the reciprocal, then multiplying by 2; i.e. applying f followed by g. So this function is gf.

(iii) This function is obtained by multiplying by 2 twice; i.e. applying g twice. So this function is g^2.

You may find the Mathcentre video ***Composition of functions*** useful.

Inverse functions

Example 3
The function f is defined by $\mathrm{f}(x) = 2x^3 + 1$. Find the inverse function f^{-1}.

Solution

The function can be written as: $y = 2x^3 + 1$

Interchanging x and y: $x = 2y^3 + 1$

Rearranging: $x - 1 = 2y^3$

$$\frac{x-1}{2} = y^3$$
$$y = \sqrt[3]{\frac{x-1}{2}}$$

The inverse function is $\mathrm{f}^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$.

Remember that to for a function to have an inverse function, it must be one-to-one (you can find the inverse of a many-to-one mapping, but this would be a one-to-many mapping, which is not a function).

Example 4

The functions f and g are defined as:

$$f(x) = x^2 - 4 \quad x \geq 0$$

$$g(x) = \sqrt{x-3} \quad x \geq 3$$

(i) What is the range of each function?
(ii) Find the inverse function f^{-1}, stating its domain.
(iii) Find the inverse function g^{-1}, stating its domain.
(iv) Write down the range of f^{-1} and the range of g^{-1}.

Solution

(i) The range of f is $f(x) \geq -4$.
The range of g is $g(x) \geq 0$.

(ii) The function can be written as $y = x^2 - 4$

Interchanging x and y: $x = y^2 - 4$

Rearranging: $x + 4 = y^2$

$y = \sqrt{x+4}$

The domain of f^{-1} is the same as the range of f.

$f^{-1}(x) = \sqrt{x+4} \qquad x \geq -4$

(iii) The function can be written as $y = \sqrt{x-3}$

Interchanging x and y: $x = \sqrt{y-3}$

Rearranging: $x^2 = y - 3$

$y = x^2 + 3$

The domain of g^{-1} is the same as the range of g.

$g^{-1}(x) = x^2 + 3 \qquad x \geq 0$

(iv) The range of f^{-1} is $f^{-1}(x) \geq 0$ (the same as the domain of f)
The range of g^{-1} is $g^{-1}(x) \geq 3$ (the same as the domain of g)

You can investigate inverse functions and their graphs using the Geogebra resource ***Inverse functions***.

You may also find the Mathcentre video ***Inverse functions*** useful.