Edexcel Core 4 Parametric equations

Section 1: Using parametric equations

Notes and Examples

These notes contain subsections on

- Parametric equations
- Sketching a parametric curve
- Finding the cartesian equation of a curve
- Parametric equations involving trigonometric functions
- Finding areas

Parametric equations

An equation like $y = 5x + 1$ or $y = 3\sin x + 4\cos x$ or $x^2 + y^2 = 1$ is called a **cartesian equation**. A cartesian equation gives a direct relationship between x and y.

In **parametric equations** x and y are both defined in terms of a third variable (**parameter**) usually t or θ.

For example $x = t, \quad y = t^2$ is a pair of **parametric equations**
and $x = \cos\theta, \quad y = \sin\theta$ is also a pair of **parametric equations**.

Parametric equations can be used for a complicated curve which doesn't have a simple Cartesian equation.

Sketching a parametric curve

You won't be asked to sketch parametric curves in your examination. However, it is useful to do a few, as this gives you a feel for how parametric curves work.

To sketch a curve given its parametric equations follow these steps.

Step 1 Make a table like this one:

t or θ				
x				
y				

Step 2 Choose values of t or θ (these will be usually be given to you)

Step 3 Work out the corresponding values of x and y using the parametric equations.

Step 4 Plot the (x, y) coordinates. Join them up in a smooth curve.

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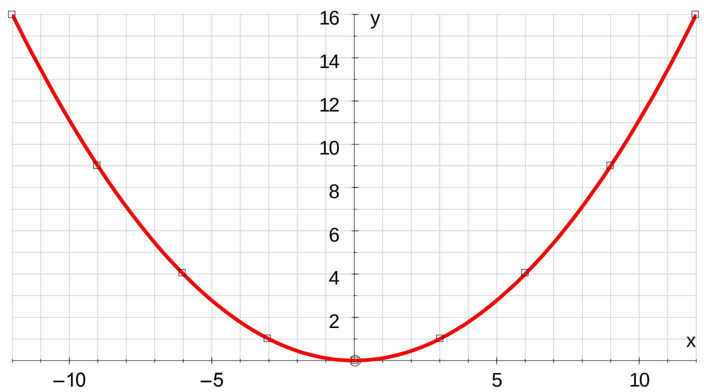
These examples show you how to do this.

Example 1
A curve has the parametric equations $x = 3t, y = t^2$.
Sketch the curve for $-4 \le t \le 4$

Solution

t	-4	-3	-2	-1	0	1	2	3	4
x	-12	-9	-6	-3	0	3	6	9	12
y	16	9	4	1	0	1	4	9	16



Finding the cartesian equation of a curve

To find the cartesian equation of a curve from its parametric equations you need to eliminate the parameter.

Note: it is not always possible to find the cartesian equation of a curve defined parametrically.

Step 1 Make *t* the subject of one of the parametric equations.
Step 2 Substitute your equation for *t* into the other parametric equation.
Step 3 Simplify.

Example 2
Find the cartesian equation of the curve defined by the parametric equations
$x = 3t + 2,\ y = 1 - t^2$

Solution

Make *t* the subject of $x = 3t + 2$: $\qquad t = \dfrac{x-2}{3}$ ①

Substitute ① into $y = 1 - t^2$: $\qquad y = 1 - \left(\dfrac{x-2}{3}\right)^2$

Simplify:
$$y = 1 - \left(\frac{x-2}{3}\right)^2$$
$$y = 1 - \frac{(x-2)^2}{9}$$

Sometimes it is easier to rearrange both equations to give an expression for t, as shown in the next example.

Example 3

Find the cartesian equation of the curve defined by the parametric equations

$x = \frac{t}{t+1}, \; y = \frac{t}{t-1}$

Solution

$$x = \frac{t}{t+1} \Rightarrow x(t+1) = t \Rightarrow tx + x = t \Rightarrow t - tx = x \Rightarrow t(1-x) = x \Rightarrow t = \frac{x}{1-x}$$

$$y = \frac{t}{t-1} \Rightarrow y(t-1) = t \Rightarrow ty - y = t \Rightarrow ty - t = y \Rightarrow t(y-1) = y \Rightarrow t = \frac{y}{y-1}$$

Equating the two expressions for t:
$$\frac{x}{1-x} = \frac{y}{y-1}$$
$$x(y-1) = y(1-x)$$
$$xy - x = y - xy$$
$$2xy - y = x$$
$$y(2x-1) = x$$
$$y = \frac{x}{2x-1}$$

Parametric equations involving trigonometric functions

Parametric equations often involve trigonometric functions.

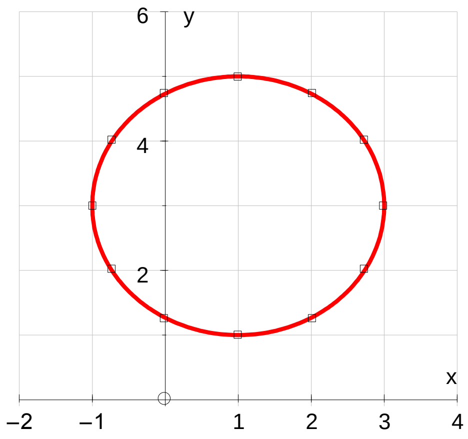
Example 4

A curve has the parametric equations $x = 1 + 2\cos\theta, y = 3 + 2\sin\theta$.

Sketch the curve for $0° \le \theta \le 360°$.

Solution

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
x	3	2.73	2	1	0	-0.73	-1	-0.73	0	1	2	2.73	3
y	3	4	4.73	5	4.73	4	3	2	1.27	1	1.27	2	3



If you want to find the Cartesian equation for parametric equations involving trigonometric functions, you will probably need to use a trigonometric identity.

You might need to use any of the Pythagorean identities:

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

Alternatively, you might need the double angle or compound angle formulae.

Example 5

Find the cartesian equation of the curve defined by the parametric equations

$x = 4 + \cos\theta, \quad y = 2\cos 2\theta$

Solution

The identity which connects $\cos 2\theta$ and $\cos\theta$ is

$$\cos 2\theta \equiv 2\cos^2\theta - 1 \qquad ①$$

$$x = 4 + \cos\theta \Rightarrow \cos\theta = x - 4 \qquad ②$$

$$y = 2\cos 2\theta \Rightarrow \cos 2\theta = \frac{y}{2} \qquad ③$$

Substitute ② and ③ into ①

$$\frac{y}{2} \equiv 2(x-4)^2 - 1$$

Simplify $\qquad y \equiv 4(x-4)^2 - 2$

A circle with radius r and centre $(0, 0)$ has parametric equations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

A circle with radius r and centre (a, b) has parametric equations:

$$x = a + r\cos\theta$$
$$y = b + r\sin\theta$$

Example 6

(i) Find the cartesian equation of the curve given by the parametric equations

$$x = 2 + 3\cos\theta$$
$$y = 3\sin\theta - 1$$

(ii) Sketch the curve.

Solution

The trig identity $\sin^2\theta + \cos^2\theta \equiv 1$ links $\sin\theta$ and $\cos\theta$

Make $\cos\theta$ the subject of $x = 2 + 3\cos\theta$: $\cos\theta = \dfrac{x-2}{3}$ ①

Make $\sin\theta$ the subject of $y = 3\sin\theta - 1$: $\sin\theta = \dfrac{y+1}{3}$ ②

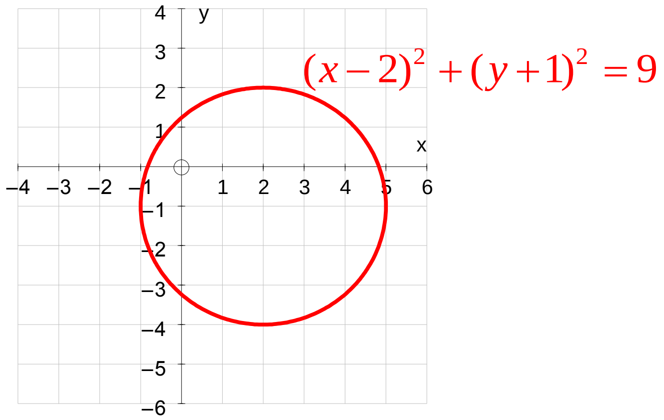
Substitute ① and ② into $\sin^2\theta + \cos^2\theta \equiv 1$:

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

So $$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{9} = 1$$

So $$(x-2)^2 + (y+1)^2 = 9$$

(ii) Need to sketch a circle of radius 3 and centre (2, -1)



An ellipse has parametric equations of the form $x = a\cos\theta,\ y = b\sin\theta$.

Try recreating the ***Parametric Equations Pictures***, either on your own or with a group of friends.

Finding areas

In Core 2 you learnt to find areas using integration.

The area under a curve from $x = a$ to $x = b$ is given by $\int_a^b y\,\mathrm{d}x$.

When working with parametric equations, you can use the chain rule so that the variable involved is the parameter:

Area $\int y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$

It is important to remember that the limits of integration must be values of t, not x.

Example 7

(i) Find the values of t at which the curve $x = 3t + 2,\ y = 1 - t^2$ meets the x-axis.

(ii) Find the area enclosed between the curve and the x-axis.

Solution

(i) When the curve meets the x-axis, $y = 0 \Rightarrow 1 - t^2 = 0 \quad \Rightarrow t = \pm 1$

(ii) $x = 3t + 2 \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 3$

$$\begin{aligned}
\text{Area } &= \int_{-1}^{1} y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{-1}^{1} (1 - t^2) \times 3\,\mathrm{d}t \\
&= \int_{-1}^{1} (3 - 3t^2)\,\mathrm{d}t \\
&= \left[3t - t^3\right]_{-1}^{1} \\
&= (3 - 1) - (-3 + 1) \\
&= 4
\end{aligned}$$