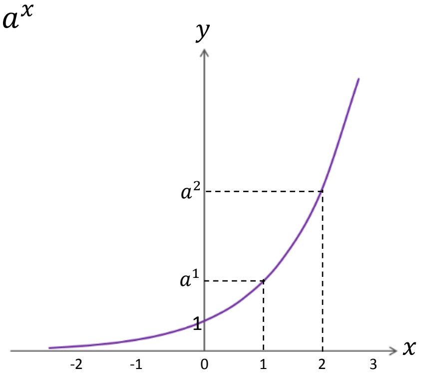
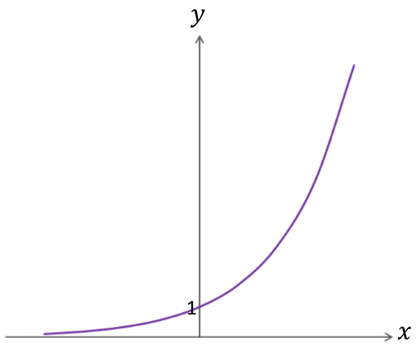
Summary sheet: Exponentials and logarithms

F1 Know and use the function a^x and its graph, where a is positive
Know and use the function e^x and its graph
F2 Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications
F3 Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$
Know and use the function ln x and its graph
Know and use ln x as the inverse function of e^x
F4 Understand and use the laws of logarithms: $\log_a x + \log_a y = \log_a(xy)$, $\log_a x - \log_a y = \log_a(x/y)$, $k\log_a x = \log_a x^k$ (including for example k = -1 and k = ½)
F5 Solve equations of the form $a^x = b$
F6 Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y
F7 Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models

Exponential Functions

Graph:	Points to notice	Tips
$y = a^x$ 	• Always crosses the y-axis at 1 ($a^0 = 1$) • The x-axis is an asymptote as you can never get a y-value of 0 ($a^x \neq 0$)	If you don't remember what the graph looks like, try substituting a with a number (e.g. use 3^x) and find some points. Plot them to get an idea of what the graph looks like.
$y = \mathrm{e}^x$ 	• The graph looks the same as you have just replaced a with e.	Find a few points to see what the graph looks like.

The gradient of e^{kx}

If: $y = \mathrm{e}^{kx}$

then: $\frac{\mathrm{d}y}{\mathrm{d}x} = k\mathrm{e}^{kx}$

Remember that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the gradient function and so this is the gradient of e^{kx}

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Logs

What does a log mean? E.g. $\log_2 16$

This is a log with base 2.

$\log_2 16$ means "***What power do I raise 2 to, to get 16?***" (i.e. $2^? = 16$)
The answer is 4 (because $2^4 = 16$)

$\therefore \log_2 16 = 4$

You can summarise like this:

$\boldsymbol{y = \log_a x \iff x = a^y}$ (for $a > 0$ and $x > 0$)

This means that "log to the base n" and "n to the power of" are the opposite (inverse) of each other and will undo each other (cancel each other out).

e.g. $2^{\log_2 5} = 5$ **and** $\log_2(2^5) = 5$

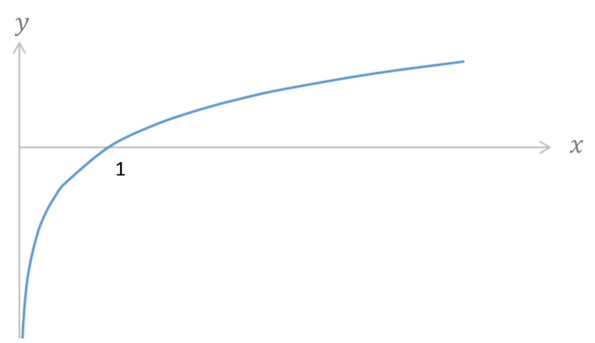
Cancel each other out   Cancel each other out

ln (the natural log)

ln has a base of e, and so ln and e are the opposite (inverse) of each other and will undo each other (cancel each other out).

e.g. $\mathrm{e}^{\ln 7} = 7$ **and** $\ln(e^7) = 7$

The graph of $\ln x$

Graph:	Points to notice
$y = \ln x$  	• Always crosses the x-axis at 1 ($\ln 1 = 0$) • The y-axis is an asymptote as you cannot get an answers for $\ln 0$ (try it on your calculator, you will get an error - you can't raise e to any power and get the answer 0)

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The laws of logs

You need to learn, and know how to use, the following laws of logs:

Law	Example
$\mathbf{\log(x) + \log(y) = \log(xy)}$	$\log(2) + \log(5) = \log(2 \times 5) = \log(10)$
$\mathbf{\log(x) - \log(y) = \log\left(\frac{x}{y}\right)}$	$\log(12) - \log(3) = \log\left(\frac{12}{3}\right) = \log(4)$
$\mathbf{\log(x^k) = k\log(x)}$	$\log(5^2) = 2\log(5)$
$\mathbf{\log(1) = 0}$	$\log_{27} 1 = 0$

All of the laws are true for any base (including base e, i.e. ln).

Solve equations of the form $a^x = b$

To solve this type of equation you need to bring the x down from the power, so you will use the 3rd law:

$$\mathbf{\log(x^k) = k\log(x)}$$

Step 1: Take the log of both sides.
Step 2: use the 3rd rule to bring the power to the front.
Step 3: Solve the equation as normal.

e.g. Solve the equation: $3^{x-5} = 2$

Step 1: Take the log of both sides: $\log(3^{x-5}) = \log(2)$

Step 2: use the 3rd rule: $(x-5)\log 3 = \log 2$

Step 3: Tidy up and solve:

$$(x-5) = \frac{\log 2}{\log 3}$$

$$(x-5) = 0.6309$$

$$x = 0.6309 + 5$$

$$\mathbf{x = 5.6309}$$

Summary sheet: Exponentials and logarithms

Logarithmic Graphs

When you have a relationship of the form $y = kx^n$ or $y = ab^x$ it can be tricky to find the parameters $(k, a \text{ and } b)$ from the curve. Taking logs of both sides turns the relationship into a straight line and makes finding the parameters easier.

	$y = kx^n$	$y = ab^x$
Original:	$y = kx^n$	$y = ab^x$
Take logs of both sides:	$\log(y) = \log(kx^n)$	$\log(y) = \log(ab^x)$
Tidy up using laws of logs:	$\log(y) = \log(k) + \log(x^n)$	$\log(y) = \log(a) + \log(b^x)$
	$\log(y) = \log(k) + n\log(x)$	$\log(y) = \log(a) + x\log(b)$
	$\log(y) = n\log(x) + \log(k)$	$\log(y) = x\log(b) + \log(a)$
You now have a straight line (of the form $y = mx + c$) where:	gradient $= n$ intercept $= \log k$	gradient $= \log b$ intercept $= \log a$

For either of the above you can plot the graph and find the gradient and the intercept.

e.g. you have been given data for x and y and it is thought that the relationship is of the form $y = kx^n$. Verify this and find the approximate values of k and n.

Data:

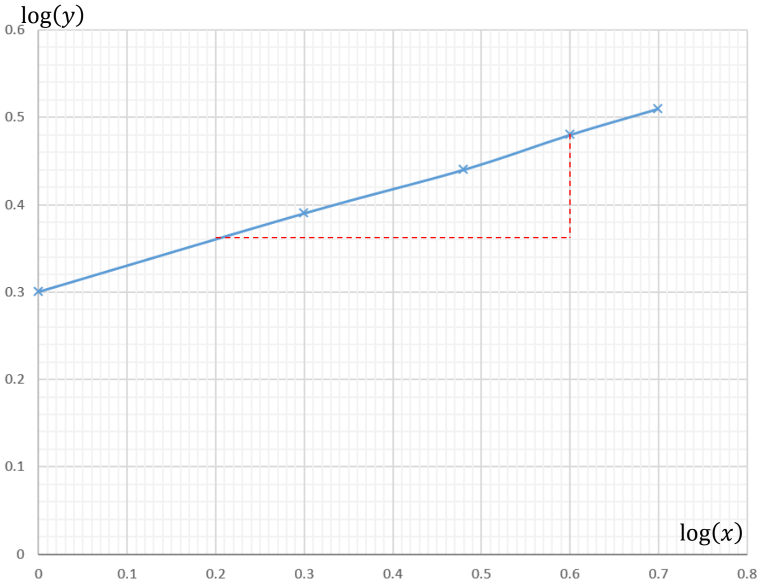
x	1	2	3	4	5
y	2	2.46	2.78	3.03	3.24

Take logs of each side, as shown above, to get: $\log(y) = n\log(x) + \log(k)$

You need to plot $\log(y)$ against $\log(x)$ so first of all find the values of $\log(y)$ and $\log(x)$:

x	1	2	3	4	5
$\log(x)$	0	0.3	0.48	0.6	0.7
y	2	2.46	2.78	3.03	3.24
$\log(y)$	0.3	0.39	0.44	0.48	0.51

Now plot the graph of $\log(y)$ against $\log(x)$:



Gradient $(n) = \frac{0.48-0.36}{0.6-0.2} = 0.3$

Intercept $(\log(k)) = 0.3$

$\therefore k = 10^{0.3} = 1.99 \ (\approx 2)$

You have found that the relationship is approximately:

$y = 2x^{0.3}$

Summary sheet: Exponentials and logarithms

e.g. you have been given data for x and y and it is thought that the relationship is of the form $y = ab^x$. Verify this and find the approximate values of a and b.

Data:

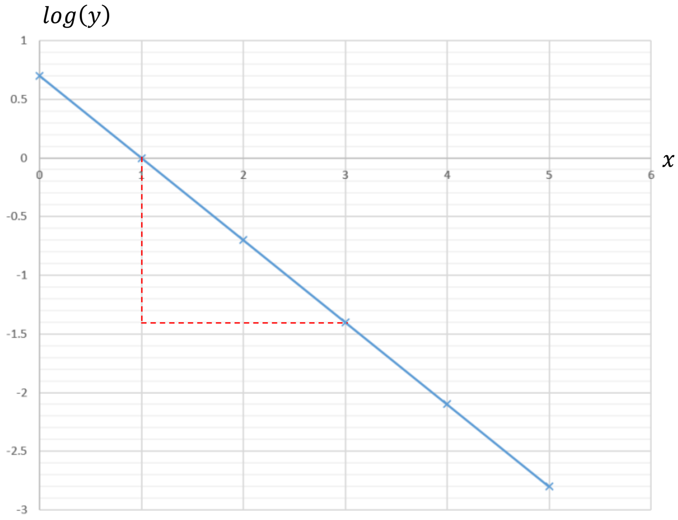
x	0	1	2	3	4	5
y	5	1	0.2	0.04	0.008	0.0016

Take logs of each side, as shown above, to get: $\log(y) = x\log(b) + \log(a)$

You need to plot $\log(y)$ against x so first of all find the values of $\log(y)$:

x	0	1	2	3	4	5
y	5	1	0.2	0.04	0.008	0.0016
$\log(y)$	0.7	0	-0.7	-1.4	-2.1	-2.8

Now plot the graph of $\log(y)$ against x:



Gradient $(\log(b)) = \frac{-1.4-0}{3-1} = -0.7$

$\therefore b = 10^{-0.7} \approx 0.2$

Intercept $(\log(a)) = 0.7$

$\therefore a = 10^{0.7} = 5.01\ (\approx 5)$

You have found that the relationship is approximately:

$$y = 5 \times 0.2^x$$