Ritangle - an A Level Maths Competition 2016

Questions and Answers - A-E, 1-22

Note that the answers here combine to help in solving the Final Question (Question 23; this question is not given here).

The Taster Questions

A. In the below r, i, t, a, n, g, l, and e are non-zero real numbers.

A sequence is defined as follows:

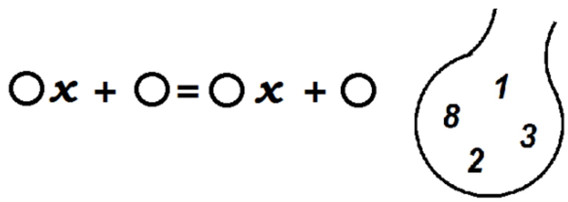
$$u_1 = r,\ u_2 = i,\ u_3 = t,\ u_4 = a,\ u_5 = n,\ u_6 = g,\ u_7 = l,\ u_8 = e.$$

Subsequent terms are defined as $\dfrac{1}{\text{product of previous eight terms}}$.

What is u_{100}?

Answer to A: this sequence cycles. The first eight terms are, r, i, t, a, n, g, l, e, while the ninth is $\dfrac{1}{ritangle}$. The tenth goes back to r, and so every nine terms, the sequence repeats itself. Thus after 99 terms, the next must be r.

B. Take the four numbers from the bag below and put them into the circles in some order (no repeats!)



How many different equations can you make? What are the solutions? Write down the possible positive integer solutions in descending order.

Answer to *B*: we can write down four numbers in 24 different orders, but $ax + b = cx + d$ is the same equation as $cx + d = ax + b$, so we in fact get 12 different equations here with twelve different solutions. Only three of these are positive whole numbers $(7, 5$ and $3)$, so the answer is 753.

C. A triangle has angles in degrees that are all integers. One is a square, another is a cube and the third is a fourth power. Write down the sizes of the three angles in descending order.

Answer to C: We can see with a spreadsheet that $10^2 + 4^3 + 2^4 = 180$ is the only possibility that works.

The angles are therefore $100°, 64°$ and $16°$, and answer is 1006416.

D. Given a positive integer n, we say $s(n)$ is the sum of all the factors of n not including n itself.

Thus $s(6) = 1+2+3 = 6, s(7) = 1, s(8) = 1+2+4 = 7, s(9) = 1+3 = 4$.

It is easy to find even numbers n so that $s(n) > n$, for example $s(12) = 1 + 2 + 3 + 4 + 6 = 16$.

It's harder to find odd numbers where $s(n) > n$, but it is possible; for example, $s(1575) = 1649 > 1575$.

There is one odd number smaller than 1575 so that $s(n) > n$; what is it?

Answer to D: the secret to having a large value for $s(n)$ is for n to have lots of small prime factors. So if we factorise 1575, we get $1575 = 3^2 \times 5^2 \times 7$.

So we can try values for p and q in $3^p \times 5^q \times 7$ perhaps. It transpires that $945 = 3^3 \times 5 \times 7$, and $s(945) = 975$.

So 945 is the answer.

E. Replace the question marks truthfully below using the whole numbers from 5 to 29 inclusive (no repeats!). Then write down the three red question-mark numbers in descending order.

? and ? are numbers with first digit 2, that add to 50.

3 and ? are the prime factors of ?

? is the square of ?

? and ? are twin primes.

The number of Archimedean solids is ?, which is half ?

? is both an odd number and a cube.

? > ? are each one more than a Fibonacci number,

and one less than a triangle number.

? and ? and ? and ? multiply to 73370.

? and ? have an HCF that is one less than ?

? + ? = 20, and their LCM is ?

? and ? multiply to 1 less than an odd square.

5 6 7 8 9 10 11 12 13 14 15 16 17

18 19 20 21 22 23 24 25 26 27 28 29

Answer to *E*: It's easy to look up that there are 13 Archimedean solids, which deals with 13 and 26. The number 27 is the only odd cube here. The numbers 9 and 14 are the only possible answers for the Fibonacci/Triangle numbers clue. The number 73370 factorises into $2 \times 5 \times 11 \times 23 \times 29$. So this gives us $10, 11, 23$ and 29, or $5, 22, 23$ and 29. The numbers 22 and 28 are the only possibilities left for the top clue. The third clue must use 5 and 25. Thus $5, 22, 23$ and 29 is impossible, and we have $10, 11, 23$ and 29. The numbers 19 and 17 are now the prime pair, and the only possible answers to the last three clues are $15, 20, 6$ and $8, 12, 24$, and $16, 18$. The three red question marks are $6, 5$ and 9, so 965 is the answer.

The Main Puzzle Questions with Answers

1. The equation of the perpendicular bisector of the line AB, where $A = (2, 5)$ and $B = (6, 3)$ is what?

Answer to 1: the midpoint of AB is $(4, 4)$. The gradient of AB is $-\frac{1}{2}$, so a line perpendicular to this has gradient 2. So the perpendicular bisector of AB is of the form $y = 2x + k$, and since $(4, 4)$ is on this line, k is -4.

2. Take a positive integer x, cube each of its its digits and add the results together to get a positive integer y. Now do the same to y, to get a positive integer z. If $x = 1$, then $z = x$. What the next value for x so that $z = x$? *Note: this value is less than 1000.*

Answer to 2: we can use a spreadsheet here, and programming is not needed. We take a three-digit number (including 0 as a digit), split it into its digits, cube and add them, then do the same again and compare. Finding the first digit of a three digit is easy using the command 'round-down', which rounds down to the nearest integer. So if we have the number $A1 = 123$, rounddown$(A1/100, 0)$ gives us $1 = A2$ (the 0 means, 'to 0 decimal places'). Writing rounddown$((A1 - 100 * A2)/10, 0)$ gives $2 = A3$. Then $A1 - 100 * A2 - 10 * A3$ gives us $3 = A4$.

We can now check in when the final two columns first match after 1. The answer is 136.

3. A right-angled triangle has integer sides length x, y and z where $x < y < z$. Adding the three side lengths gives 810, while multiplying the three side lengths gives 13284 times this. What's the area of the triangle?

Answer to 3: we have $xyz = 10760040, x+y+z = 810$, and $x^2+y^2 = z^2$. Let A be the area we seek, and so $2A = xy$. We have $z = 810 - x - y$, and $x^2+y^2 = (810-x-y)^2$, which on multiplying out, cancelling and rearranging gives $x+y = \dfrac{810^2+4A}{2 \times 810}$. We also have $2A(810-(x+y)) = 810 \times 13284$. Substituting in for $x+y$ gives us a quadratic in A;

$$4A^2 - 810^2 A + 810^2 \times 13284.$$

Solving this gives $A = 149445$ (which leads to complex values for x and y) or $A = 14580$. This happens for x=81 or 360, which means $x = 81, y = 360$, and $z = 369$. So our answer is 14 580.

An alternative is to draw the graphs of $xy(810-x-y) = 10760040$ and $x^2+y^2 = (810-x-y)^2$ and see where they cross.

Note; it is also possible to use Excel on this problem.

4. The line $y = mx + k$ touches the parabola $y = ax^2 + bx + c$ (where $a \neq 0$) at the point (p, q). If $m = 8a^2 + 4ab + 12ac + b$, what is p (in terms of a, b and c)?

Answer to 4: The line touches the parabola only if the gradient of the curve and the line are equal. Thus on differentiating, $2ap + b = 8a^2 + b(4a + 1) + 12ac$, and so $p = 4a + 2b + 6c$.

If we don't want to differentiate, we can ask, where does the line meet the curve? Where $mx + k = ax^2 + bx + c$, or $ax^2 + x(b - m) + (c - k) = 0$. We know that the line touches the curve, so the discriminant must be zero, giving $(b-m)^2 = 4a(c-k)$. We know $q = mp+k$, and $q = ap^2+bp+c$, so $k = ap^2+bp+c-mp$. Substituting in, we get $(b-m)^2 = 4a(c-(ap^2+bp+c-mp))$, and if we multiply out and factorise, we get $(2ap + b - m)^2 = 0$. Now substituting in for m, we get our result.

Note; using a CAS facility (like the one in Geogebra) is a real help here.

5. An arithmetic progression has third term $32j + 19k$ and tenth term $18j + 12k$. What, in terms of j and k, is the sixteenth term?

Answer to 5: if a is the first term of the arithmetic sequence, then $a+2d = 32j+19k$, and $a+9d = 18j+12k$, so subtracting these we have $-7d = 14j+7k$, or $d = -2j - k$. This means that $a = 36j + 21k$, and so the sixteenth term $= a + 15d = 36j + 21k - 30j - 15k = 6j + 6k$.

6. How many solutions (to three significant figures) does the equation (in radians) $\sin(10^9 x) = 0.1$ have in the interval $0 \leq x \leq 325$?

Answer to 6: We have $\sin(10^9 x) = 0.1 \implies 10^9 x = 0.1001... + 2n\pi$ OR $\pi - 0.1001... + 2n\pi$. This tells that

$$x = \frac{\arcsin 0.1}{10^9} + \frac{2n\pi}{10^9} \quad \text{OR} \quad \frac{\pi - \arcsin 0.1}{10^9} + \frac{2n\pi}{10^9}.$$

Thus we have two solutions for every $\frac{2\pi}{10^9}$, so we have approximately $\frac{325 \times 10^9}{\pi}$ solutions, which is 1.03×10^{11}.

7. If $\dfrac{x^5}{y^2} = 98304$, and $\dfrac{y^5}{x^2} = \dfrac{6561}{64}$, then what is x?

Answer to 7: x and y must both be positive. We have $\dfrac{x^5}{y^2} = 98304 \implies y = \sqrt{\dfrac{x^5}{98304}}$. Thus $\dfrac{\left(\sqrt{\dfrac{x^5}{98304}}\right)^5}{x^2} = \dfrac{6561}{64} \implies x^{25/2-2} \times 98304^{-5/2} = \dfrac{6561}{64} \implies x^{10.5} = \dfrac{6561 \times 98304^{2.5}}{64} \implies x = 24$.

Alternatively we can multiply our two equations to get $(xy)^3 = 10077696$, and so $xy = 216$. Now substituting $y = \dfrac{216}{x}$ into our first equation gives $x = 24$.

Alternatively we can draw the curves $\dfrac{x^5}{y^2} = 98304$, and $\dfrac{y^5}{x^2} = \dfrac{6561}{64}$, and see where they cross.

8. We can say $\cos x + \sin x$ is is one of these, while $\cos x + \sin(\pi x)$ is not one of these, but $\cos^2 x + \sin^2 x$ is one of these, although $\cos(x^2) + \sin(x^2)$ is not one of these, however, $\cos x \sin x$ is one of these...

Answer to 8: We could try drawing the graphs of these functions. It is noticeable that the curves that 'are one of these' each repeat in a regular fashion, while the others don't. We say that such graphs represent **periodic functions**.

9. The polynomial $ax^3+bx^2+cx-68000$ gives a remainder of 6000 when divided by $x-1$, a remainder of 5000 when divided by $x-2$, and a remainder of 4000 when divided by $x-3$. What's the remainder when we divide by $x-4$?

Answer to 9: Using the remainder theorem, we find that $a+b+c-68000=6000, 8a+4b+2c-68000=5000, 27a+9b+3c-68000=4000$. We can solve these (a computer algebra system is a big help here) to find $a=12500, b=-75000, c=136500$. Now we need the value of $12500x^3-75000x^2+136500x-68000$ when $x=4$, which is 78000.

10. *Make a hat charm? I may, I may not* (anagram). When did I die?

Answer to 10: this is *Omar Khayyam, mathematician.* He died in 1131.

11. The Indian mathematician Ramanujan famously pointed out that the number 1729 was special, since $1729 = 1^3 + 12^3 = 9^3 + 10^3$. The value 1729 is in fact the smallest that can be written as the sum of two positive cubes in two different ways.

What's the smallest number that can be written as the sum of a positive cube and a fourth power in two different ways? The answer is $4097 = 1^3 + 8^4 = 16^3 + 1^4$. *We can debate as to how different these ways actually are!* What's the next smallest such number?

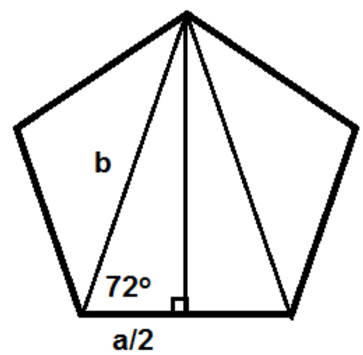
Answer to 11: we need computer help here. We can create a spreadsheet, with values of x running horizontally and values of y vertically, and with $x^3 + y^4$ in the cells; can we find repeats?

Or else we could run a program like this in Excel:

```
Private Sub CommandButton1_Click()
k = 40
n = Cells(40, 1)
For a = 1 To n
For b = 1 To n
For c = 1 To n
For d = 1 To n
If (a ^ 3 + b ^ 4) = (c ^ 3 + d ^ 4) And (a <> c) Then
k = k + 1
Cells(k, 1) = a
Cells(k, 2) = b
Cells(k, 3) = c
Cells(k, 4) = d
Cells(k, 5) = (a ^ 3 + b ^ 4)
End If
Cells(40, 2) = a
Next d
Next c
Next b
Next a
End Sub
```

If we choose $n = 25$ (choosing n too big means the program cannot cope), it gives us what we need; $9^3 + 10^4 = 22^3 + 3^4 = 10729$.

12. If a regular pentagon has sides of length a and diagonals of length b, then $\frac{b}{a}$ is what?



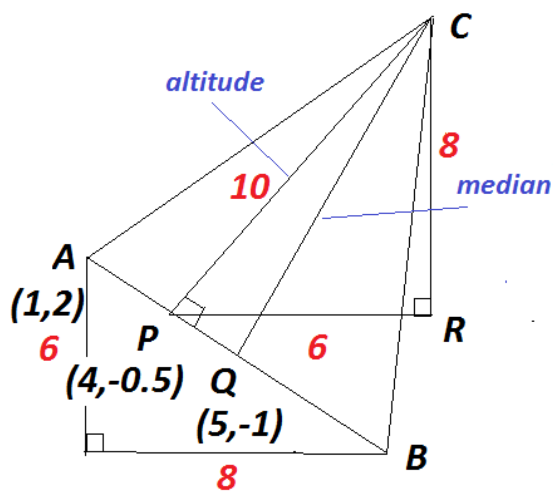
Answer to 12: We have $\frac{b}{a} = \frac{1}{2\cos 72^\circ} = 1.618...$, which googling tells us is a famous number called ϕ. Like the numbers π and e, ϕ has a long history in mathematics. It is also known as **the golden ratio**.

13. What's $\dfrac{(6a^2+9ab+3b^2)(6a^2-8ab+2b^2)}{(a^2-b^2)(18a-6b)}$ when simplified?

Answer to 13: The top factorises to $6(a+b)(a-b)(2a+b)(3a-b)$, while the bottom factorises to $6(a+b)(a-b)(3a-b)$, so after cancelling we get $2a+b$.

14. In a triangle ABC, A, P, Q and B are collinear. A is the point $(1,2), P = (4,-0.25)$ is the foot of the altitude from C to AB, and $Q = (5,-1)$ is the foot of the median from C to AB. The length PC is the same as AB. The point C is in the first quadrant. Find the coordinates of C, and multiply them together.

Answer to 14: the points A and Q tell us B is at $(9,-4)$. Thus AB is $10 = PC$.



We can see we have two $3-4-5$ triangles in the diagram, in particular PRC. This gives us that $C = (10, 7.75)$, and so our answer is $10 \times 7.75 \times 150 = 11625$.

15. A triangle has two sides of length $\sqrt{380}$ and $2+\sqrt{95}$. The angle between them is $60°$. What's the length of the third side?

Answer to 15: we can use the cos rule here. If the missing side is x, then $x^2 = (2+\sqrt{95})^2 + 380 - 2(2+\sqrt{95})\sqrt{380}\cos(60°)$. Using the fact that $380 = 4 \times 95$, multiplying out gives us that $x = 17$.

16. If $a = 0.25631$, what's the largest coefficient of any power of x (to three significant figures) in the expansion of

$$(x + a)^{100}?$$

Answer to 16: this is a question where we need computing power, like that offered by Derive. This can expand $(x + 0.25631)^{100}$ for us instantaneously. If we check along, the highest coefficient of any power of x here is 8.025319435×10^8 (the coefficient of $x^8 0$). Thus our answer is 803 000 000.

We can alternatively use Excel here to accomplish much the same thing. We need to find out how to write the binomial coefficients in Excel; once we've done that, the problem is straightforward.

17. What's number 47 in this sequence?

1. Given a line segment AB, it's possible to construct an equilateral triangle with AB as one of the sides.
2. Given a line segment AB and a point C, its possible to construct a line segment CD so that the lengths of AB and CD are equal.
3. ...

Answer to 17; 1. and 2. are the first two of Euclid's theorems in his seminal collection of thirteen books, *The Elements*. Number 47 on his list is Pythagoras' Theorem, so that is our answer.

18. The expression $(((3x^3+7x+1)^2+(1-x)^5)^5+((3x^2-12)^3+(9x^3-x)^5)^2)^2$ is a what?

Answer to 18: this must be a polynomial in x. Multiplying out shows that 60 is the highest power, so this is a **degree sixty polynomial**.

19. If A is the point $(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + ..., 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ...)$ and B is the point $(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ..., 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...)$, then -5 represents the what?

Answer to 19: we have here four geometric progressions each with $|r| < 1$, and so they have a sum to infinity, which in each case is $\dfrac{a}{1-r}$. So A is the point $\left(\dfrac{4}{5}, \dfrac{4}{3}\right)$, while B is the point $\left(\dfrac{2}{3}, 2\right)$. So the gradient of AB is -5, and our answer is the **gradient of AB**.

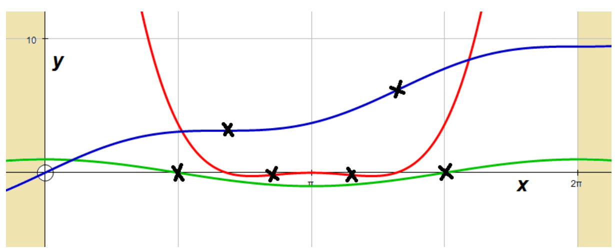
20. Over the interval $0 < x < 2\pi$, what do these three curves all have?

1. $y = (x-\pi)^4 - (x-\pi)^2$,

2. $y = \dfrac{3x}{2} + \sin\left(\dfrac{3x}{2}\right)$,

3. $y = \cos x$.

Answer to 20: we are looking for the fact that they all have two points of inflection. A point of inflection happens when the gradient function has a turning point. We can spot a point of inflection since it's where the tangent to the curve moves from one side of the curve to the other. The points of inflexion are marked in below.



Thus our answer is **two points of inflection**.

21. As x varies, what's the minimum value of $y = 2x^2 - 12ax - 16bx + 18a^2 + 48ab + 2a + 32b^2 - 3b$?

Answer to 21; we could differentiate and put the gradient equal to zero. Or we could complete the square:

$$2x^2 - 12ax - 16bx + 18a^2 + 48ab + 2a + 32b^2 - 3b = 2(x - (3a + 4b))^2 + 2a - 3b,$$

so the minimum possible value for y is $2a - 3b$.

22. Did he use his mathematical these to solve an age-old puzzle?

Answer to 22: the age-old problem that was Fermat's Last Theorem succumbed in 1997 to Andrew Wiles, the English mathematician then based in Princeton. The story of how Professor Wiles found a way through the proof is extraordinary; a beautiful account appears in Simon Singh's book *Fermat's Last Theorem*. So our answer here is **wiles**.