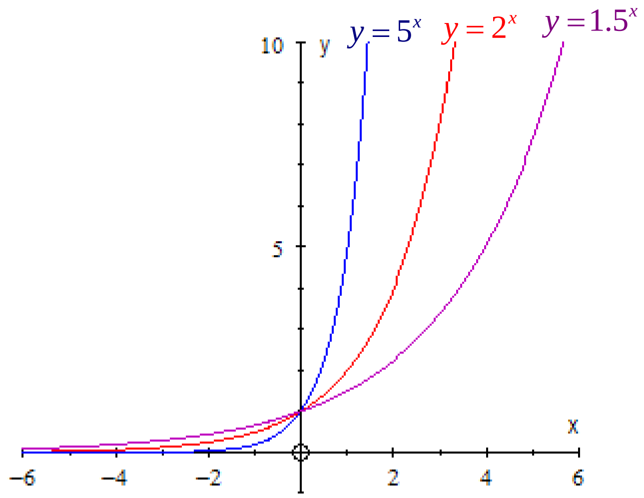
Exponentials and logarithms
Section 1: General exponential functions and logarithms
Notes and Examples

These notes contain subsections on

Exponential functions

An exponential function is any function of the form $y = a^x$. The graphs below show some different exponential functions.

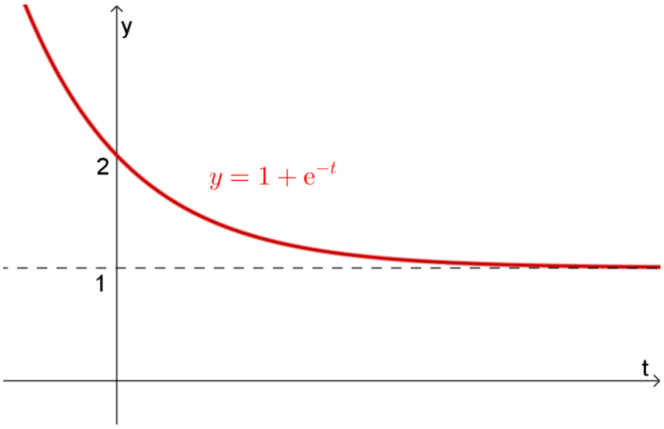


Many real life situations can be modelled by exponential functions. The growth of a population (e.g. of people, animals or bacteria) can be modelled by an exponential function. A model like this might take the form $y = c \times a^{kt}$. This type of model is called **exponential growth**.

In an exponential growth model, the quantity being modelled continues to increase, at an ever-increasing rate. In a real-life situation such as the growth of a population, the model will eventually break down, since other factors such as overcrowding or limited resources will affect the growth of the population.

Another type of model is **exponential decay**, in which something decreases exponentially. A model like this might take the form $y = c \times a^{-kt}$. Exponential decay could model the temperature of a cooling liquid, or the mass of a radioactive isotope remaining.

In an exponential decay model, the quantity being modelled decreases at a rate which becomes slower and slower. The quantity will approach a limiting value, but never quite reach it. For example, the graph below shows the curve $y = 1 + e^{-t}$. The graph approaches the line $y = 1$ as t becomes large.



Indices and logarithms

Logarithms are the inverse of exponentials.

The important thing to remember about logarithms is that, although they appear to be a new topic, they are simply about writing what you already know about indices in a different way.

If you find it difficult to work out the meaning of a statement involving logarithms, it can be simpler to change the statement into the equivalent statement involving indices.

$$\log_a b = x \Leftrightarrow a^x = b.$$

To remember this, notice that a is both the base of the logarithm and the base of the index, and x, the logarithm, is the index. The value of $\log_a b$ is the answer to the question: "What power must I raise a to in order to get b?"

Example 1

(i) Find $\log_4 2$

(ii) Find x, where $\log_5 x = -\frac{1}{2}$

Solution

(i) The statement $\log_4 2 = x$ is equivalent to $4^x = 2$.

Since $4^{\frac{1}{2}} = 2$, then x must be $\dfrac{1}{2}$.

So $\log_4 2 = \frac{1}{2}$.

(ii) The statement $\log_5 x = -\tfrac{1}{2}$ is equivalent to $5^{-\frac{1}{2}} = x$.

So $x = \dfrac{1}{\sqrt{5}}$.

The laws of logarithms

The laws of logarithms are:

$$\log x + \log y = \log xy$$

$$\log x - \log y = \log \frac{x}{y}$$

$$\log x^n = n \log x$$

It's a worthwhile exercise to try to work through these proofs

These can be proved using the laws of indices:

First convert into index notation: $\log_c x = a \Leftrightarrow c^a = x$

$$\log_c y = b \Leftrightarrow c^b = y$$

To prove the first law:

Using the laws of indices

$$c^a c^b = xy \Leftrightarrow c^{a+b} = xy$$

$$\Leftrightarrow \log_c xy = a + b$$

$$\Leftrightarrow \log_c xy = \log_c x + \log_c y$$

Similarly for the second law:

$$\frac{c^a}{c^b} = \frac{x}{y} \Leftrightarrow c^{a-b} = \frac{x}{y}$$

$$\Leftrightarrow \log_c \frac{x}{y} = a - b$$

$$\Leftrightarrow \log_c \frac{x}{y} = \log_c x - \log_c y$$

For the third law:

$$\log_c x^n = a \Leftrightarrow c^a = x^n$$

$$\Leftrightarrow c^{a/n} = x$$

$$\Leftrightarrow \log_c x = \frac{a}{n}$$

$$\Leftrightarrow n \log_c x = a$$

$$\Leftrightarrow n \log_c x = \log_c x^n$$

As the first two laws of indices require the indices to have the same base, then the first two laws of logarithms require the logarithms to have the same base.

Example 2

(i) Write $\log \frac{x^3 y}{\sqrt{z}}$ in terms of log *x*, log *y* and log *z*.

(ii) Write $2\log a - \log b - \frac{1}{3}\log c$ as a single logarithm.

Solution

(i) $$\log \frac{x^3 y}{\sqrt{z}} = \log x^3 + \log y - \log\sqrt{z}$$
$$= \log x^3 + \log y - \log z^{1/2}$$
$$= 3\log x + \log y - \tfrac{1}{2}\log z$$

(ii) $$2\log a - \log b - \tfrac{1}{3}\log c = \log a^2 - \log b - \log c^{1/3}$$
$$= \log a^2 - (\log b + \log\sqrt[3]{c})$$
$$= \log \frac{a^2}{b\sqrt[3]{c}}$$

Solving exponential equations using logarithms

Many equations are solved using inverse functions, for example you solve the equation $x + 3 = 5$, in which addition is applied to the unknown variable, by subtracting 3 from each side. Similarly you solve the equation $x^2 = 10$ by using the square root function, which is the inverse of the square function.

Exponential functions are the inverse of logarithm functions: the function $y = a^x$ is the inverse of the function $y = \log_a x$. An equation like $2^x = 10$ involves an exponential function of x. So to solve this equation, it follows that you need to use the inverse of the exponential function, which is the logarithm function. This is shown in the next example.

Example 3

Solve the following equations.

(i) $2^x = 10$

(ii) $3^{2x-1} = 4$

(iii) $0.2^{1-x} = 2$

Solution

(i) $$2^x = 10$$
$$\log 2^x = \log 10$$
$$x\log 2 = \log 10$$
$$x = \frac{\log 10}{\log 2} = 3.32$$

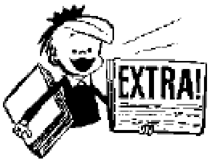
(ii) $3^{2x-1}=4$

$$\log 3^{2x-1}=\log 4$$
$$(2x-1)\log 3=\log 4$$
$$2x-1=\frac{\log 4}{\log 3}$$
$$x=\frac{1}{2}\left(\frac{\log 4}{\log 3}+1\right)=1.13$$

(iii) $0.2^{1-x}=2$

$$\log 0.2^{1-x}=\log 2$$
$$(1-x)\log 0.2=\log 2$$
$$1-x=\frac{\log 2}{\log 0.2}$$
$$x=1-\frac{\log 2}{\log 0.2}=1.43$$

An old practical application of logarithms



Before calculators existed, logarithms were used to make calculations easier. For example, suppose you had to divide 1432627 by 967253. You could do this by long division, but it would take a long time and the chances of making a mistake would be quite high. So you would apply the second law of logarithms:

$$\log(1432627 \div 967253) = \log 1432627 - \log 967253$$

To do the calculation, you would have to find the log to base 10 of the two numbers, subtract the results, and then find the inverse log of the answer.

You would have to find the values of $\log 1432627$ and $\log 967253$ from a book of tables. Unfortunately most tables would only tell you the values of $\log x$ for values of x between 10 and 99. So you would then use the fact that

$$\begin{aligned}\log\ 1432627 &= \log(14.32627\times 100000)\\ &= \log 14.32627+\log 100000\\ &= \log 14.32627+5\end{aligned}$$

You would then use the tables to find the value of $\log 14.33$ (which is as accurate as most tables would give you). This would give a value for $\log 1432627$ of 6.1562.

You would then go through a similar process to find $\log 967253$.

$$\begin{aligned}\log 967253 &= \log 96.7253+\log 10000\\ &= 1.9855+4\\ &= 5.9855\end{aligned}$$

Next you would subtract these two logarithms (without a calculator of course!), giving 0.1707.

Now you would have to find the number whose logarithm is 0.1707. Inverse log tables usually give values between 1 and 10.

$$\begin{aligned} 0.1707 &= 1.1707 - 1 \\ &= \log 14.81 - \log 10 \\ &= \log \frac{14.81}{10} \\ &= \log 1.481 \end{aligned}$$

So $1432627 \div 967253 = 1.481$.

Most pupils did not understand the theory behind these calculations; they just followed a set of instructions to use the tables of logarithms and work out the calculation. Even after calculators became widely available, it was several years before this technique was removed from examination syllabuses!

Logarithms were also the basis of slide rules, which were also used before calculators existed to work out calculations quickly.

