Edexcel AS Mathematics: Trigonometry

Section 2: Trigonometric equations

Notes and Examples

These notes contain the following subsections:

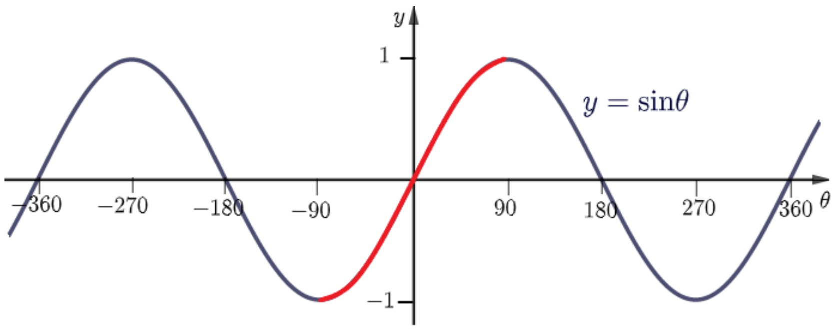
Principal values

There are infinitely many solutions to an equation like $\sin\theta = \frac{1}{2}$.

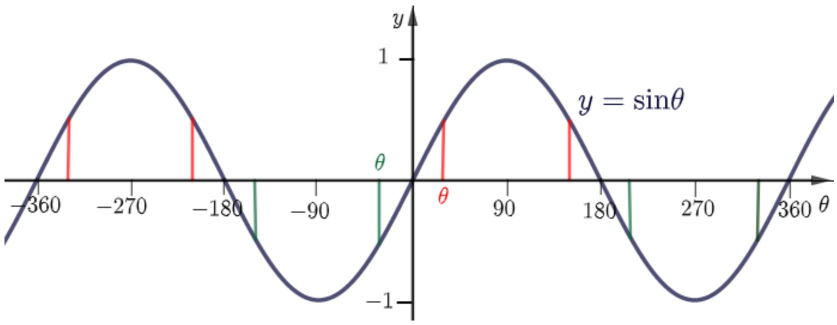
Your calculator will only give one solution – the **principal value**.

You find this by pressing the calculator keys for $\arcsin\frac{1}{2}$ (or $\sin^{-1}\frac{1}{2}$ or $\text{inv}\sin\frac{1}{2}$). Check that you can get the answer of 30°.

For $\sin\theta$ the principal value is between −90° and 90°.

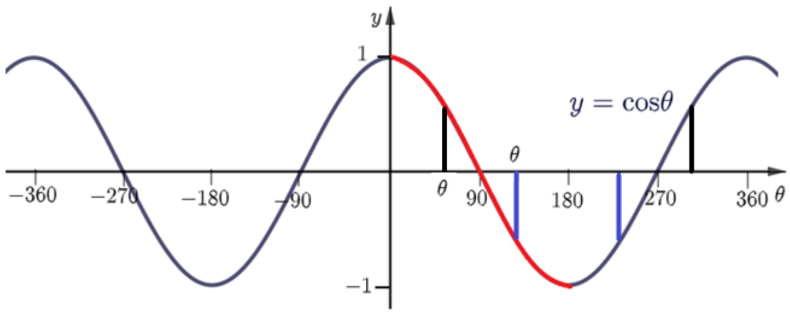


You can use symmetry to find other solutions.



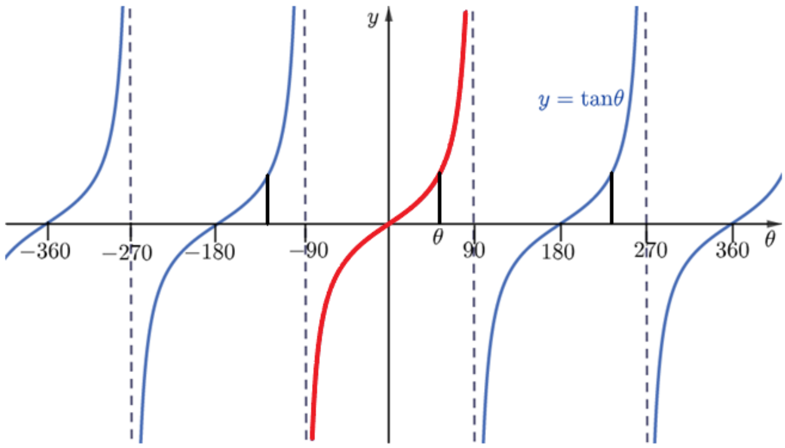
A second solution in a 360° cycle can be found by $180° - \theta$.

When you use the inverse cosine function, your calculator will always give you an answer from 0° to 180°.



A second solution in a 360° cycle can be found by $360° - \theta$

When you use the inverse tan function, your calculator will always give you an answer between −90° and 90°.



A second solution in a 360° cycle can be found by $\theta + 180°$ or $\theta - 180°$.

Alternatively, you can use the CAST diagram to find other solutions.

Solving simple trigonometric equations

Because there are infinitely many solutions to a trigonometric equation you are only ever asked to find a few of them! Any question at this level asking you to solve a trigonometric equation will also give you the interval or range of values in which the solutions must lie. For example, you might be asked to solve for $0° \le \theta \le 360°$.

You can only directly solve trigonometric equations like $\sin\theta = \frac{1}{2}$ or $\cos\theta = \frac{1}{4}$ or $\tan\theta = -2$. Here is an example.

Example 1

Solve $\sin\theta = \frac{\sqrt{3}}{2}$ for $-360° \le \theta \le 360°$.

Solution

$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60°$

There will be a second solution in the second quadrant.

$180° - 60° = 120°$ is also a solution.

Since $y = \sin\theta$ has a period of 360° any other solution can be found by adding/subtracting 360° to these two solutions.

So the other solutions are:

$$60° - 360° = -300°$$

and $$120° - 360° = -240°$$

So the values of θ for which $\sin\theta = \frac{\sqrt{3}}{2}$ are $-300°, -240°, 60°, 120°$.

Almost all equations of the type in Example 1 have two solutions in the range $0° \le \theta \le 360°$.

However, this is not true of all trigonometric equations. Suppose, for example, that you want to solve the equation $\sin 2x = 0.5$ in the range $0° \le \theta \le 360°$. You can find that $2x = 30°$ or $150°$, which means that $x = 15°$ or $75°$. However, there are two further solutions in the range $0° \le \theta \le 360°$, given by $2x = 30° + 360° = 390° \Rightarrow x = 195°$, and $2x = 150° + 360° = 510° \Rightarrow x = 255°$. So there are four solutions: $x = 15°, 75°, 195°, 255°$.

This means that if you are solving an equation of the form $\sin nx = k$, you need to adjust the range in which you look for initial solutions.

Example 2

Solve each of the following equations in the given range:

(a) $\tan 3x = 1$ for $0° \le \theta \le 360°$

(b) $\cos(2x + 40°) = \frac{1}{2}$ for $-180° < x \le 180°$

Solution

(a) $\tan 3x = 1$

You need to look for solutions for $3x$ in the range 0° to 1080°

$$3x = 45°,\ 225°,\ 405°,\ 585°,\ 765°,\ 945°$$
$$x = 15°,\ 75°,\ 135°,\ 195°,\ 255°,\ 315°$$

(b) $\cos(2x + 40°) = \frac{1}{2}$

The lower end of the range you need to look in is $-180° \times 2 + 40° = -320°$, and the upper end is $180° \times 2 + 40° = 400°$.

$$2x + 40° = -300°,\ -60°,\ 60°,\ 300°$$
$$2x = -340°,\ -100°,\ 20°,\ 260°$$
$$x = -170°,\ -50°,\ 10°,\ 130°$$

More complicated trigonometric equations

Any more complicated equations need to be manipulated algebraically before they can be solved. There are a number of techniques you can use:

- Rearrange the equation to make $\sin\theta, \cos\theta$ or $\tan\theta$ the subject.
- Check to see if the equation factorises to give two (or more) equations which involve just one trigonometric function (see Example 3).
- If it is a quadratic in either $\sin\theta, \cos\theta$ or $\tan\theta$ it can either be factorised or solved using the formula for solving quadratic equations (see Example 4).
- If the equation involves just $\sin\theta$ and $\cos\theta$ (and no powers), check to see if you can use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (see Example 5).
- If the equation contains a mixture of trigonometric functions (e.g. $\cos^2\theta$ and $\sin\theta$) then you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to make it a quadratic in either $\sin\theta, \cos\theta$ or $\tan\theta$ (see Example 6).

Example 3

Solve $2\cos\theta\sin\theta + \cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

$2\cos\theta\sin\theta + \cos\theta = 0$ can be factorised as there is $\cos\theta$ in both terms on the LHS.

It is wrong to divide through by $\cos\theta$ because you lose the solutions to $\cos\theta = 0$.

Factorising gives: $\cos\theta(2\sin\theta + 1) = 0$

So either $\cos\theta = 0$ or $2\sin\theta + 1 = 0$

$\cos\theta = 0 \Rightarrow \theta = 90°$

By symmetry, $360° - 90° = 270°$ is also a solution.

$$2\sin\theta + 1 = 0$$
$$\Rightarrow \quad \sin\theta = -\tfrac{1}{2}$$
$$\Rightarrow \quad \theta = -30°$$

So, $-30° + 360° = 330°$ is a solution in the right range.

By symmetry, $180° - (-30°) = 210°$ is also a solution.

So the values of θ for which $2\cos\theta\sin\theta + \cos\theta = 0$ are 90°, 210°, 270° and 330°.

In Example 4 you need to solve a quadratic equation.

Example 4

Solve $2\cos^2\theta + 3\cos\theta = 2$ for $0° \le \theta \le 360°$.

Solution

$2\cos^2\theta + 3\cos\theta = 2$ is a quadratic equation in $\cos\theta$.

Rearrange the quadratic: $2\cos^2\theta + 3\cos\theta - 2 = 0$

You can replace $\cos\theta$ with x to make things simpler!
Or factorise straightaway to get $(2\cos\theta - 1)(\cos\theta + 2) = 0$ and then solve.

Let $x = \cos\theta$: $2x^2 + 3x - 2 = 0$

Factorising gives: $(2x - 1)(x + 2) = 0$

So, $x = \tfrac{1}{2} \Rightarrow \cos\theta = \tfrac{1}{2}$
or $x = -2 \Rightarrow \cos\theta = -2$

$\cos\theta = -2$ has no real solutions.

So you need to solve $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$

By symmetry, $360° - 60° = 300°$ is also a solution

So the values of θ for which $2\cos^2\theta + 3\cos\theta = 2$ are 60° and 300°.

In the next example you need to use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$.

Example 5

Solve $\sin\theta - 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

You need to rearrange the equation: $\sin\theta - 2\cos\theta = 0$

> You can safely divide by $\cos\theta$ because it can't be equal to 0. If it were then $\sin\theta$ would also have to be 0 (as you would have $\sin\theta - 2\times 0 = 0$) and $\cos\theta$ and $\sin\theta$ are never both 0 for the same value of θ.

Dividing by $\cos\theta$: $\dfrac{\sin\theta}{\cos\theta} - 2 = 0$

Since $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$: $\tan\theta - 2 = 0$

$\Rightarrow \tan\theta = 2$

$\Rightarrow \theta = 63.4°$ to 1 d.p.

By symmetry, $63.4° + 180° = 243.4°$ is also a solution.

So the values of θ for which $\sin\theta - 2\cos\theta = 0$ are 63.4° and 243.4° to 1 d.p.

> In general, you should give angles correct to 1 decimal place.

In the next example you need to use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 6

Solve $\sin^2 x + \sin x = \cos^2 x$ for $0° \le \theta \le 360°$.

Solution

Rearranging the identity $\sin^2 x + \cos^2 x \equiv 1$

gives: $\cos^2 x \equiv 1 - \sin^2 x$ ①

Substituting ① into the equation $\sin^2 x + \sin x = \cos^2 x$ gives:

$$\sin^2 x + \sin x = 1 - \sin^2 x$$

This is a quadratic in $\sin x$.

Rearranging give: $2\sin^2 x + \sin x - 1 = 0$

This factorises to give: $(2\sin x - 1)(\sin x + 1) = 0$

So either: $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ $\Rightarrow \sin x = -1$

$\Rightarrow x = 30° \text{ or } 150°$ $\Rightarrow x = 270°$

So the solutions to $\sin^2 x + \sin x = \cos^2 x$ are $x = 30°,\ 150°,\ 270°$